AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: All Units	**Free Response Question Stem Types** **Tabular**	**Date:** April 27, 2020

Free Response Questions Stem Types: Tabular

2020 FRQ Practice Problem BC1

x	1	2	3	4	5
$f'(x)$	62	30	20	15	12

BC1: The function f is twice differentiable for $x \geq 1$ where $f(5) = -6$. Selected values of the positive and decreasing function f', the derivative of f, are given in the table above. The graph of f' has horizontal asymptote $y = 0$.

The series $\sum_{n=1}^{\infty} a_n$ is defined where $a_n = f'(n)$.

(**a**) Use a right Riemann sum with the four subintervals indicated in the table to approximate $f(1)$. Is this approximation an over or under estimate of $f(1)$? Give a reason for your answer.

$$f(5)-f(1)=\int_1^5 f'(x)\,dx \Rightarrow f(1)=f(5)-\int_1^5 f'(x)\,dx \Rightarrow f(1)=(-6)-\int_1^5 f'(x)\,dx$$

$$f(1)\approx(-6)-\left[(30)(1)+(20)(1)+(15)(1)+(12)(1)\right]=(-6)-\left[(30)+(20)+(15)+(12)\right]$$

$$=(-6)-[77]=-83 \qquad f(1)\approx-83$$

This estimate is an overestimate because $f'(x)$ is a decreasing function and the Reimann sum of 77 is an underestimate so we have subtracted too little.

(**b**) Evaluate $\int_1^{\infty} f''(x)dx$.

$$\int_1^{\infty} f''(x)\,dx = \lim_{b\to\infty}\int_1^b f''(x)\,dx = \lim_{b\to\infty}\left[f'(x)\right]_1^b = \lim_{b\to\infty}\left[f'(b)-f'(1)\right] = \lim_{b\to\infty} f'(b)-62$$

$$=\lim_{b\to\infty} f'(b)-62=-62 \qquad \text{horizontal asymptote} \Rightarrow \lim_{b\to\infty} f'(b)=0$$

The problem has been restated.

x	1	2	3	4	5
$f'(x)$	62	30	20	15	12

BC1: The function f is twice differentiable for $x \geq 1$ where $f(5) = -6$. Selected values of the positive and decreasing function f', the derivative of f, are given in the table above. The graph of f' has horizontal asymptote $y = 0$.

The series $\sum_{n=1}^{\infty} a_n$ is defined where $a_n = f'(n)$.

(**c**) If $\int_1^5 x f''(x)dx = -100$, find $f(1)$.

$$\int_1^5 x\, f''(x)\, dx$$

$$\int \underbrace{x}_{u}\, \underbrace{f''(x)\, dx}_{dv} \qquad u = x \Rightarrow du = dx \qquad dv = f''(x)dx \Rightarrow v = f'(x)$$

$$= \left[x\, f'(x)\right]_1^5 - \int_1^5 f'(x)\, dx \qquad = x\, f'(x) - \int f'(x)\, dx$$

$$= \left[(5)f'(5) - (1)f'(1)\right] - \left[f(5) - f(1)\right] = \left[(5)(12) - (1)(62)\right] - \left[(-6) - f(1)\right]$$

$$= \left[(60) - (62) + 6\right] + f(1) = 4 + f(1) = -100 \Rightarrow f(1) = -104$$

(**d**) Determine if the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges or diverges. Explain your reasoning.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \sum_{n=1}^{\infty} \left[(-1)^{n+1} f'(n)\right] = 62 - 30 + 20 - 15 + 12 \cdots$$

$(-1)^{n+1} a_n$ are alternating $\qquad a_n = f'(n)$ is decreasing

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f'(n) = 0$ because f' has a horizontal asymptote $y = 0$

By the Alternating Series Test $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

(**e**) Consider the series $\sum_{n=1}^{\infty} b_n$ where $(1 + 2^{a_n})b_{n+1} = (a_n)!\, b_n$. Use the ratio test to determine if the series $\sum_{n=1}^{\infty} b_n$ converges or diverges.

$$\left(1 + 2^{a_n}\right) b_{n+1} = (a_n)!\, b_n \Rightarrow \frac{b_{n+1}}{b_n} = \frac{(a_n)!}{\left(1 + 2^{a_n}\right)}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f'(n) = 0$ because f' has a horizontal asymptote $y = 0$

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{(a_n)!}{\left(1 + 2^{a_n}\right)}\right| = \frac{(0)!}{1 + 2^0} = \frac{1}{2} < 1 \Rightarrow \text{converges by the Ratio Test.}$$

Free Response Questions Stem Types: Tabular
2020 FRQ Practice Problem BC2

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

BC2: The functions f and g are twice differentiable for all values of x. Selected values of f, g and their derivatives f' and g' are given in the table above.

(**a**) Let $y = Q(t)$ be the particular solution to the logistic differential equation $\frac{dy}{dt} = 3y(g(4) - y)$. Find the rate when $Q(t)$ is increasing the fastest.

$\frac{dy}{dx} = 0 \Rightarrow y = 0$ or $g(4) - y = 0 \Rightarrow y = g(4) = 10$

$y = 10$ is the carrying capacity or horizontal asymptote L

greatest rate of change is when $y = \frac{L}{2} = 5$ $\quad \left.\frac{dy}{dx}\right|_{y=5} = 3(5)(g(4) - 5) = 3(5)(10 - 5) = 75$

The rate when $Q(t)$ is increasing the fastest is 75.

A large grocery store, customers are entering and exiting the checkout lines. At time $t = 4$ minutes, there are 84 people waiting in line so the manager decides to open up several more checkout lanes. For $4 \le t \le 8$ minutes, the rate that customers enter a check out line is given by $f'(t)$ and the rate that customers exit a check out line is given by $g'(t)$ where f' and g' are measured in people per minute.

(**b**) Is the number of customers in line increasing or decreasing at time $t = 4$ minutes?

Let $P(t)$ be the amount of people in line.

$P'(t) = f'(t) - g'(t) \qquad P'(4) = f'(4) - g'(4) = 1 - 4 = -3$

The amount of people in line is decreasing at time $t = 4$ minutes because $P'(4) < 0$.

(**c**) Find the number of customers in line at time $t = 8$ minutes.

$$P(8) = P(4) + \int_4^8 P'(t)\,dt = 84 + \int_4^8 \left[f'(t) - g'(t)\right]dt = 84 + \left[f(t) - g(t)\right]_4^8$$

$$= 84 + \left[f(8) - g(8)\right] - \left[f(4) - g(4)\right] = 84 + \left[(7) - (16)\right] - \left[(0) - (10)\right] = 85$$

The problem has been restated.

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

(**d**) Let $s(x) = \dfrac{g(x)}{3x}$. Find $s'(2)$.

$$s'(x) = \frac{(3x)g'(x) - 3g(x)}{(3x)^2} \Rightarrow s'(2) = \frac{(6)g'(2) - 3g(2)}{(6)^2} = \frac{(6)5 - 3(4)}{36} = \frac{18}{36} = \frac{1}{2}$$

Let $H(x) = 3x + \int_1^{x^2} g(x)\,dx$.

(**e**) Find $H'(2)$ and $H''(2)$.

$$H'(x) = 3 + g(x^2)(2x) \Rightarrow H'(2) = 3 + g(4)(4) = 3 + (10)(4) = 43$$
$$H''(x) = g(x^2)(2) + g'(x^2)(2x)(2x)$$
$$H''(2) = g(4)(2) + g'(4)(4)(4) = (10)(2) + (4)(4)(4) = 20 + 64 = 84$$

(**f**) Find the second degree Taylor polynomial for $H(x)$ centered at $x = -1$.

$$H(-1) = 3(-1) + \int_1^{(-1)^2} g(x) = -3 \qquad H'(-1) = 3 + g(1)(-2) = 3 + (5)(-2) = -7$$
$$H''(x) = g(x^2)(2) + g'(x^2)(2x)(2x)$$
$$H''(-1) = g(1)(2) + g'(1)(-2)^2 = (5)(2) + (-4)(-2)^2 = 10 - 16 = -6$$
$$P_2(x) = H(-1) + H'(-1)(x+1) + \frac{H''(-1)}{2!}(x+1)^2$$
$$P_2(x) = (-3) + (-7)(x+1) + \frac{(-6)}{2!}(x+1)^2 = (-3) - 7(x+1) - 3(x+1)^2$$

Free Response Questions Stem Types: Tabular

2020 FRQ Practice Problem BC3

x	1	2	3	4	5
$f'(x)$	62	30	20	15	12

BC3: The function f is twice differentiable for $x \geq 1$ where $f(5) = -6$. Selected values of the positive and decreasing function f', the derivative of f, are given in the table above. The graph of f' has horizontal asymptote $y = 0$.

The series $\sum_{n=1}^{\infty} a_n$ is defined where $a_n = f'(n)$.

(**a**) Evaluate $\int_{-1}^{0} f''(1-3x)dx$.

$$\int_{-1}^{0}\int_{-1}^{0} f''(1-3x)\ \ dx = \int_{-1}^{0} f''(1-3x)(-3xdx)\ \ dx = -\frac{1}{3}\int_{-1}^{0} f''\underbrace{(1-3x)}_{u}\underbrace{(-3dx)}_{du}$$

$$= -\frac{1}{3}\left[f'(1-3x)\right]_{-1}^{0} = -\frac{1}{3}\left[f'(1-3(0)) - f'(1-3(-1))\right] = -\frac{1}{3}\left[f'(1) - f'(4)\right]$$

$$= -\frac{1}{3}\left[(62)-(15)\right] = -\frac{47}{3}$$

(**b**) Evaluate $\int_{5}^{\infty} f''(x)\sin\left(f'(x)\right)dx$.

$$\int_{5}^{\infty} f''(x)\sin\left(f'(x)\right)dx = \lim_{b\to\infty}\int_{5}^{b} f''(x)\sin\left(f'(x)\right)dx = \lim_{b\to\infty}\int_{5}^{b} \sin\underbrace{\left(f'(x)\right)}_{u}\underbrace{\left(f''(x)dx\right)}_{du}$$

$$= \lim_{b\to\infty}\left[-\cos f'(x)\right]_{5}^{b} = \lim_{b\to\infty}\left[-\cos f'(b) + \cos f'(5)\right] = \lim_{b\to\infty}\left[-\cos f'(b) + \cos(12)\right]$$

$$= -\lim_{b\to\infty}\cos f'(b) + \cos(12) = -\cos(0) + \cos(12) = -1 + \cos(12)$$

$$\text{horizontal asymptote} \Rightarrow \lim_{b\to\infty} f'(b) = 0$$

(**c**) Write an equation of the line tangent to $f(x)$ at $x = 5$. Use the tangent line to approximate $f(5.1)$.

$$T(x) = f(5) + f'(5)(x-5) = (-6) + 12(x-5)$$

$$f(5.1) \approx T(5.1) = (-6) + 12(5.1-5) = (-6) + 12(0.1) = (-6) + (1.2) = -4.8$$

The problem has been restated.

x	1	2	3	4	5
$f'(x)$	62	30	20	15	12

BC3: The function f is twice differentiable for $x \geq 1$ where $f(5) = -6$. Selected values of the positive and decreasing function f', the derivative of f, are given in the table above. The graph of f' has horizontal asymptote $y = 0$.

The series $\sum_{n=1}^{\infty} a_n$ is defined where $a_n = f'(n)$.

(**d**) Use Euler's method, starting at $x = 5$ with two steps of equal size, to approximate $f(3)$.

x	$f(x)$	$dy = f'(x)dx$
5	-6	$(12)(-1) = -12$
4	-18	$(15)(-1) = -15$
3	-33	

(**e**) Use a left Riemann sum with the four subintervals indicated in the table to approximate the arc length of $f(x)$ over the interval $x = 1$ to $x = 5$.

$$L = \int_1^5 \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

$$\approx \left[\sqrt{1+\left[f'(1)\right]^2}(1) + \sqrt{1+\left[f'(2)\right]^2}(1) + \sqrt{1+\left[f'(3)\right]^2}(1) + \sqrt{1+\left[f'(4)\right]^2}(1)\right]$$

$$= \sqrt{1+(62)^2} + \sqrt{1+(30)^2} + \sqrt{1+(20)^2} + \sqrt{1+(15)^2}$$

(**f**) For $x \geq 6, f'(x) = \frac{100}{2^x}$. Find $\sum_{n=6}^{\infty} a_n$.

$$\sum_{n=6}^{\infty} a_n = \sum_{n=6}^{\infty} f'(n) = \sum_{n=6}^{\infty} \frac{100}{2^n} = \frac{100}{2^6} + \frac{100}{2^7} + \frac{100}{2^8} + \cdots \Rightarrow \text{geometric series with } r = \frac{1}{2}, a = \frac{100}{2^6}$$

$$\sum_{n=6}^{\infty} a_n = \frac{100/2^6}{1-\frac{1}{2}} = \frac{100/2^6}{\frac{1}{2}} = \frac{200}{2^6}$$

Free Response Questions Stem Types: Tabular
2020 FRQ Practice Problem BC4

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

BC4: The functions f and g are twice differentiable for all values of x. Selected values of f, g and their derivatives f' and g' are given in the table above.

(**a**) Let k be the function defined by $k(x) = \begin{cases} f'(g(x)), & x \le 1 \\ x + \cos(f(x)), & x > 1 \end{cases}$. Is k continuous at $x = 1$?

Why or why not?

$f'(g(1)) = f'(5) = 2$

$\lim_{x\to 1^-}(g(x)) = 5 \Rightarrow \lim_{u\to 5} f'(u) = 2$

$\lim_{x\to 1^+}[f(x)] = 1 \Rightarrow \lim_{u\to 1^+}[u + \cos(u)] = 1 + \cos(1)$

$k(x)$ is not continuous because $\lim_{x\to 1} k(x)$ does not exist.

(**b**) Let $h(x) = f(g(2x))$. Find $h'(1)$.

$h'(x) = f'(g(2x))(g'(2x))(2)$

$h'(1) = f'(g(2))(g'(2))(2) = f'(4)(5)(2) = 10f'(4) = 10(1) = 10$

(**c**) Find $\int_2^4 f'(2x-3)\,dx$.

$$\int_2^4 f'(2x-3)\,dx = \frac{1}{2}\left[f(2x-3)\right]_2^4 = \frac{1}{2}\left[f(8-3) - f(4-3)\right] = \frac{1}{2}\left[(1)-(-1)\right] = 1$$

The problem has been restated.

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

BC4: The functions f and g are twice differentiable for all values of x. Selected values of f, g and their derivatives f' and g' are given in the table above.

(**d**) Use Euler's method with two steps of equal size starting at $x = 1$ to approximate $f(3)$.

x	$f(x)$	$dy = f'(x)dx$
1	-1	$f'(1)(1) = -6$
2	-7	$f'(2)(1) = 0$
3	-7	

A large grocery store, customers are entering and exiting the checkout lines. At time $t = 4$ minutes, there are 84 people waiting in line so the manager decides to open up several more checkout lanes. For $4 \le t \le 8$ minutes, the rate that customers enter a check out line is given by $f'(t)$ and the rate that customers exit a check out line is given by $g'(t)$ where f' and g' are measured in people per minute.

(**e**) Approximate $f''(4.5)$. Using correct units, interpret the meaning of this value in context of the problem.

$$f''(4.5) \approx \frac{f'(5) - f'(4)}{5 - 4} = \frac{2-1}{1} = 1$$

$f''(4.5)$ is the rate at which the rate of customers entering the line is changing in people per minute per minute at $t = 4.5$ minutes.

(**f**) Is there a time t for $4 < t < 8$ such that the number of customers in line is not changing? Give a reason for your answer.

$P'(x)$ is continuous because f and g are twice differentiable so f' and g' are continuous.

$P'(4) = f'(4) - g'(4) = -3$ $\qquad$ $P'(8) = f'(8) - g'(8) = 3$

The Intermediate Value Theorem guarantees there is a number c between 4 and 8 where $P'(c) = 0$ since $P'(c) = 0$ is between $P'(4)$ and $P'(8)$.

Free Response Questions Stem Types: Tabular

2020 FRQ Practice Problem BC5

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

BC5: The functions f and g are twice differentiable for all values of x. Selected values of f, g and their derivatives f' and g' are given in the table above.

(**a**) Let $m(x) = f(x^3)$. Find $m'(2)$.

$$m'(x) = f'(x^3)(3x^2) \Rightarrow m'(2) = f'(2^3)\left(3(2)^2\right) = 4(12) = 48$$

(**b**) Evaluate $\lim\limits_{x\to5} \dfrac{g(f(x)) - x}{x^2 - 25}$.

$$\lim_{x\to5}\left[g(f(x)) - x\right] = g(f(5)) - 5 = g(1) - 5 = 5 - 5 = 0 \qquad \lim_{x\to5}\left[x^2 - 25\right] = 25 - 25 = 0$$

$$\lim_{x\to5}\frac{g(f(x)) - x}{x^2 - 25} = \underbrace{\lim_{x\to5}\frac{g'(f(x))(f'(x)) - 1}{2x}}_{\text{l'Hospital's Rule}} = \frac{g'(f(5))(f'(5)) - 1}{10}$$

$$= \frac{g'(1)(2) - 1}{10} = \frac{(-4)(2) - 1}{10} = -\frac{9}{10}$$

(**c**) Find $\displaystyle\int_1^8 x g''(x)\,dx$.

$$\int_1^8 \underbrace{x}_{u}\,\underbrace{g''(x)\,dx}_{dv} = \left[x g'(x)\right]_1^8 - \int_1^8 g'(x)\,dx \qquad \begin{array}{l} u = x \Rightarrow du = dx \\ dv = g''(x)\,dx \Rightarrow v = g'(x) \end{array}$$

$$= \left[x g'(x)\right]_1^8 - \left[g(x)\right]_1^8 = \left[8g'(8) - g'(1)\right] - \left[g(8) - g(1)\right] = \left[8(1) - (-4)\right] - \left[(16) - (5)\right]$$

$$= [8 + 4] - [11] = 1$$

(**d**) Let $p(x) = g\left(f'(x)\right)$. Use a right Riemann sum with three subintervals indicated in the table to approximate $\displaystyle\int_2^8 p(x)dx$.

$$\int_2^8 p(x)\,dx \approx \left[p(4)(2) + p(5)(1) + p(8)(3)\right] + \left[g(f'(4))(2) + g(f'(5))(1) + g(f'(8))(3)\right]$$

$$= \left[g(1)(2) + g(2)(1) + g(4)(3)\right] = \left[(5)(2) + (4)(1) + (10)(3)\right] = 44$$

The problem has been restated.

x	1	2	4	5	8
$f(x)$	-1	4	0	1	7
$f'(x)$	-6	0	1	2	4
$g(x)$	5	4	10	12	16
$g'(x)$	-4	5	4	3	1

BC5: The functions f and g are twice differentiable for all values of x. Selected values of f, g and their derivatives f' and g' are given in the table above.

For $t \geq 1$, particles P and Q move along the x axis with velocities $f(t)$ and $g(t)$ respectively. At time $t = 1$, particle P is at position $x = 4$ and particle Q is at position $x = -2$.

(**e**) Use a left Riemann sum with the three subintervals indicated in the table to approximate the position of particle P at time $t = 8$.

$$P(8) = P(2) + \int_2^8 P'(t)\,dt \;= 4 + \int_2^8 f(t)\,dt \;\approx 4 + \left[f(2)(2) + f(4)(1) + f(5)(3)\right]$$

$$= 4 + \left[(4)(2) + (0)(1) + (1)(3)\right] = 4 + [11] = 15$$

$P(8) \approx 15$

(**f**) At $t = 1$, are particles P and Q moving toward or away from each other? Explain your reasoning

$P(1) = 4$ and $P'(1) = f(1) = -1 \Rightarrow$ Particle P is moving left from $x = 4$

$Q(1) = -2$ and $Q'(1) = g(1) = 5 \Rightarrow$ Particle Q is moving right from $x = -2$

The particles are moving toward each other.

(**g**) Is particle Q speeding up or slowing down at time $t = 1$? Give a reason for your answer.

$Q'(1) = g(1) = 5 \qquad Q''(1) = g'(1) = -4$

The particle Q is slowing done because the velocity $Q'(1)$ and the acceleration $Q''(1)$ have different signs.